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Crossover Behavior for a Noninteracting Disordered Electronic System in the Presence of a Weak Magnetic Field

Jian-Yang Yuan[1] **and T. R. Kirkpatrick**[1]

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Near the noninteracting localization phase transition, we investigate the crossover between orthogonal and unitary universality classes as a weak magnetic field is applied. We first derive a field-theoretic description of the noninteracting disordered electronic system in the presence of a weak magnetic field. This description contains both symmetries corresponding to the system with and without a magnetic field. We obtain a unified generating functional, from which a variety of relevant physical quantities can be calculated. The theory is then renormalized. The renormalization group flow equations contain information on the fixed points of the theory and on the crossover between the two universality classes. Within an ε ($=d-2$) expansion the crossover exponent is obtained and previous phenomenological results from the literature are recovered. We also discuss the shift in the location of the mobility edge caused by the weak field.

KEY WORDS: Localization; magnetic field; crossover.

1. INTRODUCTION

As the strength of the disorder is increased, it is well known that a disordered noninteracting electronic system undergies a localization transition if the spatial dimension is greater than two (see ref. 1 for a review). As a function of disorder, or energy, there is a mobility edge separating metallic behavior from insulating behavior. As this mobility edge is approached from the metallic side, the conductivity or diffusion coefficient vanishes continuously and if it is approached from the insulating side, the localization length diverges.[(1)]

[1] Institute for Physical Science and Technology, and Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742.
[2] Current address: McGill University, Montreal, Quebec, Canada H3A 2K6.

Wegner has mapped this phase transition problem onto an effective field theory which takes the form of a matrix nonlinear σ-model.[2] There are different universality classes, depending on the internal symmetries of the field theory. The theory can be formulated such that a system with time reversal symmetry (TRS) has a noncompact (pseudo) orthogonal symmetry and a system without TRS has a noncompact unitary symmetry. These two symmetries define the orthogonal and unitary universality classes, respectively.

In this paper we study the crossover behavior of a system near the mobility edge as a weak uniform magnetic field (breaking TRS) is applied. This field changes the universality class from orthogonal to unitary. The general problem of crossover for the localization problem has been discussed elsewhere.[3,4] The main contribution of this paper is to show how the original crossover ideas can be applied to the magnetic field problem even though the applied vector potential (which is the fundamental field) is position dependent. Earlier work on this problem was done by Opperman and Belitz[6] and Biafore *et al.*[7] Opperman and Belitz[6] used an N-orbital model with a time-reversal breaking perturbation to compute a crossover exponent ϕ and to lowest order in an ε $(=d-2)$ expansion he obtained $\phi = 2/\varepsilon$. Biafore *et al.*[7] used a Wilson–Polyakov renormalization group to obtain the same result. The main idea used in this paper is to apply field-theoretic renormalization group (RG) methods (see, e.g., ref. 8) directly on the "free energy" which depends only on the magnetic field and not on the vector potential. In this way the renormalization of a propagator that is not diagonal in momentum space is avoided.

Within a low-order ε-expansion $(\varepsilon = d-2)$ we recover the phenomenological result of Khmel'nitskii and Larkin[9] for the crossover exponent near the orthogonal fixed point. In light of a recent four-loop calculation by Wegner[10] for an easier crossover problem, it is unclear whether or not the crossover exponent of Khmel'nitskii and Larkin is exact. However, Biafore *et al.* have given a gauge invariance argument that suggests that the exact crossover exponent is trivially related to the correlation length exponent ν by $\phi = 2\nu$. In this paper we also discuss the shift in the location of the mobility edge caused by a weak magnetic field.

This paper is organized as follows. In Section 2 we set up the basic field theory such that it contains both the pseudo-orthogonal and pseudo-unitary symmetries. In Section 3 we "derive" the matrix nonlinear σ-model for the crossover problem. In Section 4 we compute the "free energy," or generating functional, using a loop expansion and dimensional regularization. In Section 5, the free energy is renormalized and the critical and crossover behaviors are discussed. In Section 6 we conclude with a discussion.

2. THE FIELD THEORY

We start with the generating functional (see, e.g., ref. 11)

$$F_n[J] \equiv -\ln \overline{Z_n[J]} \tag{2.1}$$

where n ($\to 0$) is the number of replicas, the bar denotes an average over the random potential of the system, and

$$Z_n[J] \equiv \int D\phi \exp\{-A_n[\phi, J]\} \tag{2.2}$$

$$A_n[\phi, J] = \sum_{p', p, a', a} i(S_{p+p'})^{1/2} \langle \phi_{a'p'} | \, \alpha(E - i\omega S_p - \hat{H}) \, \delta_{a'a} \delta_{p'p} - \hat{J}^{a'a}_{p'p} \, | \phi_{ap} \rangle \tag{2.3}$$

with $a = 1,..., n$; $p = 1, 2$. Here

$$J^{a'a}_{p'p}(x, y) \equiv \langle x | \, \hat{J}^{a'a}_{p'p} \, | y \rangle$$

are source fields, $S_p = i(-1)^p$, E is real, and $\omega > 0$. If the Hamiltonian $\hat{H}$ is symmetric, the field $\phi(x) \equiv \langle x \mid \phi \rangle$ can be either real or complex. If $\hat{H}$ is Hermitian but not symmetric, $\phi(x)$ must be complex. The parameter α is 1/2 for real fields and 1 for complex fields. For complex fields, $D\phi$ denotes $D(\mathrm{Re}\, \phi)\, D(\mathrm{Im}\, \phi)$. Using appropriate source fields, all quantities of physical interest can be generated. For this work, however, we can simply set $J = 0$.

At zero temperature, the impurities of the system that cause the localization can be modeled by a random potential $V(\mathbf{x})$. The distribution of the random potential is usually assumed Gaussian, i.e.,

$$P[V] = \frac{\exp[-(1/2\gamma) \int d^d x \, V^2(\mathbf{x})]}{\int DV \exp[-(1/2\gamma) \int d^d x \, V^2(\mathbf{x})]} \tag{2.4}$$

where γ measures the strength of the random potential of disorder. With a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{x})$ the Hamiltonian of the system in the coordinate representation is

$$\hat{H} = -\frac{\hbar^2}{2m} \left[\nabla + \frac{ie}{\hbar c} \mathbf{A}(\mathbf{x}) \right]^2 + V(\mathbf{x}) \tag{2.5}$$

This Hamiltonian is not symmetric. The average over the random potential yields

$$\overline{Z_n[J]} = \int D\phi \exp \left\{ - \int d^d x \, \phi^\dagger [S(\hat{H} - E + i\omega S)] \phi + \frac{\gamma}{2} \int d^d x \, [\phi^\dagger S \phi]^2 \right\} \tag{2.6}$$

and

$$\phi = \begin{pmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \\ \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{pmatrix} \tag{2.7}$$

The † sign denotes a Hermitian conjugate and the matrix S has zeros off diagonal and is equal to S_p on the diagonal.

To determine the symmetry of the problem, we examine the action at $\omega = 0$ because eventually we are interested in the long-time or zero-frequency limit. Under a linear transformation T ($\phi \to T\phi$), we have,

$$T^{\dagger} S(\hat{H} - E) T = S(\hat{H} - E) \tag{2.8}$$

Notice that the matrix $S(\hat{H} - E)$ is Hermitian, and has n positive and n negative eigenvalues for each eigenvalue of the Hamiltonian $\hat{H}$.[2] This defines the pseudo-unitarity matrix T. It follows that the theory is invariant under the pseudo-unitary $U(n, n)$ transformation of the field ϕ. $\omega \neq 0$ breaks the $U(n, n)$ symmetry into a lower symmetry, a direct product of two unitary groups $U(n) \otimes U(n)$. This can be seen by rotating ϕ_1 and ϕ_2 separately. When the magnetic field is switched off, the Hamiltonian $\hat{H}$, hence $S(\hat{H} - E)$, is symmetric. Then the imaginary parts of the complex field ϕ play exactly the same roles as the real parts do. Thus the field ϕ becomes a $1 \times 4n$ real matrix, and the Hermitian conjugate operation is reduced to a transpose operation, denoted by superscript T. Also, the invariance of the action functional under the transformation requires

$$T^{\mathrm{T}} S(\hat{H} - E) T = S(\hat{H} - E) \tag{2.9}$$

This defines the pseudo-orthogonal symmetry, $O(2n, 2n)$, of the transformation.

To obtain a bigger symmetry which simultaneously contains both symmetries, let us decompose the Hamiltonian into two parts,

$$\hat{H} = \hat{H}_R + \hat{H}_I \tag{2.10}$$

where, for any real function $f(\mathbf{x})$, $\hat{H}_R f(\mathbf{x})$ is a real function and $\hat{H}_I f(\mathbf{x})$ is a pure imaginary function. Using a matrix representation of complex number, we can write the Hamiltonian as

$$\hat{H} \equiv \begin{pmatrix} \hat{H}_R & i\hat{H}_I \\ -i\hat{H}_I & \hat{H}_R \end{pmatrix} = \hat{H}_R \otimes \tau_0 - \hat{H}_I \otimes \tau_2 \tag{2.11}$$

Here, the τ_i's are Pauli matrices. Choosing the Coulomb gauge, we have

$$\hat{H}_R = -\frac{\hbar^2}{2m}\nabla^2 + \frac{e^2}{2mc^2}A^2(\mathbf{x}) \tag{2.12a}$$

$$\hat{H}_I = -i\frac{\hbar e}{2m}\mathbf{A}(\mathbf{x})\cdot\nabla \tag{2.12b}$$

Expressing ϕ_{ap} in terms of its real part and imaginary parts,

$$\phi_{ap} = \theta_{ap} + i\psi_{ap} \tag{2.13}$$

we obtain

$$A[\Psi] = \int d^dx\, \Psi^{\mathrm{T}}[S(\hat{H} - E + i\omega S)]\,\Psi - \frac{\gamma}{2}\int d^dx\,(\Psi^{\mathrm{T}} S \Psi)^2 \tag{2.14}$$

where Ψ is a $1 \times 4n$ matrix, i.e.,

$$\Psi_{ap} = \begin{pmatrix} \theta_{ap} \\ \psi_{ap} \end{pmatrix} \tag{2.15}$$

The symmetry remains the same, as it should. In fact, it is a subgroup of $U(2n, 2n; R)$, which is isomorphic to $U(n, n; C)$, the usual pseudo-unitary group $U(n, n)$. Here, R, C, and later Q denote real, complex, and quaternion matrix elements, respectively.

Defining a set of quaternion matrices

$$\sigma_0 = \tau_0, \qquad \sigma_1 = \tau_1, \qquad \sigma_2 = i\tau_2, \qquad \sigma_3 = \tau_3 \tag{2.16}$$

we rewrite

$$\hat{H} = \hat{H}_R \otimes \sigma_0 + i\hat{H}_I \otimes \sigma_2 \tag{2.17}$$

Extending the range of $S\hat{H}$ into $2n \times 2n$ matrices with quaternion elements, we find that it satisfies Eq. (2.9) and is symmetric. The invariant transformation T has $O(n, n, Q)$ symmetry and satisfies $T^{\mathrm{T}}ST = S$. This symmetry obviously contains orthogonal and unitary symmetries. In this representation, the field Ψ is a $1 \times 2n$ matrix with each components being a 1×2

matrix. Finally, we note that in general the transformation is a function of position, i.e., $T = T(\mathbf{x}) \in O(n, n; Q)$.

For a variety of reasons, the localization phase transition problem must be formulated in terms of composite variables which are products of the ϕ fields. Physically these variables represent the fluctuations that lead to diffusive propagators, which in turn become critical at the localization transition. We first introduce $(2n)^2$ 2×2 matrices (quaternions),

$$Q_{pp'}^{aa'}; \quad a, a' = 1,..., n; \quad p, p' = 1, 2 \tag{2.18}$$

Using the identity

$$\int DQ \exp\left\{-\frac{1}{2}\int d^d x\,(\mathrm{tr}[Q^2] - 2i\gamma^{1/2}\psi^{\mathrm{T}} S^{1/2} Q S^{1/2}\psi)\right\}$$
$$= \exp\left[-\frac{\gamma}{2}\int d^d x\,(\psi^{\mathrm{T}} S \psi)^2\right] \tag{2.19}$$

and performing the Ψ integral, we find

$$\overline{Z_n[0]} = \int DQ\, e^{-A[Q]} \tag{2.20}$$

with

$$A[Q] = \tfrac{1}{2}\int d^d x\,\mathrm{tr}[Q^2] - \mathrm{Tr}[\ln(\hat{H} - E - \omega S - \gamma^{1/2} Q)] \tag{2.21}$$

This is the exact field theory in terms of the matrix Q. Similar forms of the action in different representations can be found in the literature.[7,12,13]

3. THE NONLINEAR σ-MODEL IN THE PRESENCE OF A WEAK MAGNETIC FIELD

The simplest way[14] to derive the nonlinear σ-model describing the localization transition is to expand around the saddle-point solution of the field theory defined by Eq. (2.21). The fluctuations about this solution lead to the σ-model. The saddle points are determined by the equation

$$\left.\frac{\delta A[Q]}{\delta Q}\right|_{Q=Q_s} = 0 \tag{3.1}$$

For zero magnetic field and weak disorder, we find

$$(Q_s)_{pp'} = i(-1)^p \frac{m\gamma^{1/2}}{2\hbar^2 \Gamma(d/2)} \left(\frac{mE}{2\pi\hbar^2}\right)^{(d-2)/2} \sigma_0 \delta_{pp'} \tag{3.2}$$

The magnetic field will in general shift the position of the saddle points. This is a trivial magnetic field dependence. If the magnetic field is weak, the shift is analytic and is proportional to the square of the field strength, because the saddle points do not depend on the direction of the magnetic field. For weak fields we assume that Eq. (3.2) is still valid.

Under the transformation T of the field Ψ, the matrix Q obeys the following transformation:

$$Q \to S^{-1/2}(T^{\mathrm{T}})^{-1} S^{1/2} Q S^{1/2} T^{-1} S^{-1/2} \tag{3.3}$$

Requiring Q to be invariant under this transformation, we choose Q in a manifold of the saddle point in the matrix field space, i.e., Q has the form

$$Q = S^{1/2} T Q_s T^{-1} S^{-1/2} \tag{3.4}$$

This implies that

$$\begin{aligned} Q^2 &= -\frac{m^2\gamma}{4\hbar^4 \Gamma^2(d/2)} \left(\frac{mE}{2\pi\hbar^2}\right)^{(d-2)/2} \\ Q^{\mathrm{T}} &= Q \end{aligned} \tag{3.5}$$

Equation (2.21) includes all possible fluctuations. To describe the critical behavior, however, only the long-wavelength fluctuations are needed. If we expand Eq. (2.21) around the saddle point and if we neglect constant terms and retain terms to order $(\delta Q)^2$ $(Q = Q_s + \delta Q)$ then we obtain

$$\begin{aligned} A[\delta Q] = \tfrac{1}{2} \int d^dx\, d^dy\, \mathrm{tr}[\delta(\mathbf{x}-\mathbf{y})\, \delta Q(\mathbf{y})\, \delta Q(\mathbf{x}) \\ - \gamma G(\mathbf{x}, \mathbf{y})\, \delta Q(\mathbf{y})\, G(\mathbf{y}, \mathbf{x})\, \delta Q(\mathbf{x})] - \bar{\omega} \int d^dx\, \mathrm{tr}(SQ) \end{aligned} \tag{3.6}$$

where

$$\bar{\omega} \equiv \frac{\omega}{\gamma^{1/2}}; \qquad G(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} |\, (\hat{H} - E - \gamma^{1/2} Q_s)^{-1}\, | \mathbf{y} \rangle \tag{3.7}$$

In the presence of a uniform magnetic field $B = B\hat{\mathbf{z}}$, the Green function (3.7) can be written in the quaternion representation as

$$G(\mathbf{x}, \mathbf{y}) = \tilde{G}(|\mathbf{x} - \mathbf{y}|)\, e^{\Phi(\mathbf{x}, \mathbf{y})\, \sigma_2} \tag{3.8}$$

where $\tilde{G}(|\mathbf{x}-\mathbf{y}|)$ is analytic in B, and

$$\Phi(\mathbf{x}, \mathbf{y}) = -\frac{eB}{2\hbar c}\hat{\mathbf{z}}\cdot(\mathbf{x}\times y) = -\frac{e}{\hbar c}\mathbf{A}\cdot(\mathbf{y}-\mathbf{x}) \tag{3.9}$$

Furthermore, it is easy to prove that for a function $f(x)$,

$$e^{2\Phi(\mathbf{x},\mathbf{y})\sigma_2} f(\mathbf{y}) = e^{[(\mathbf{y}-\mathbf{x})\cdot\nabla_-]} f(\mathbf{x}) \tag{3.10}$$

where

$$\nabla_\pm \equiv \sigma_0 \otimes \nabla_x \pm \sigma_2 \otimes \frac{2e}{\hbar c}\mathbf{A}(\mathbf{x}) \tag{3.11}$$

We assume that the fluctuation $\delta Q(\mathbf{x})$ is slowly varying on the scale of the mean free path and that the magnetic field is weak. Then, we can approximate $\tilde{G}(|\mathbf{x}-\mathbf{y}|)$ by the Green function without the magnetic field. For the same reason, the right-hand side of Eq. (3.10) can be formally expanded in terms of the operator $(\mathbf{y}-\mathbf{x})\cdot\nabla_-$, i.e.,

$$\begin{aligned} e^{2\Phi(\mathbf{x},\mathbf{y})\sigma_2}\delta Q(\mathbf{y}) &= \delta Q(\mathbf{x}) + (\mathbf{y}-\mathbf{x})\cdot\nabla_-\,\delta Q(\mathbf{x}) \\ &\quad + \tfrac{1}{2}(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})\colon\nabla_-\nabla_-\,\delta Q(\mathbf{x}) + \cdots \end{aligned} \tag{3.12}$$

For $\Phi = 0$ (zero magnetic field) we simply have a formal Taylor expansion of $\delta Q(\mathbf{y})$ about $\mathbf{x}$. Using that $\tilde{G}(|\mathbf{x}-\mathbf{y}|)$ is diagonal in the p index, we can factor out the $\tilde{G}$'s in Eq. (3.6) and obtain

$$\begin{aligned} A[\delta Q] = \int d^dx \Bigg\{ &\sum_{a,a'}\sum_{p,p'} \{(\delta Q^{(0)} + \delta Q^{(2)})^{aa'}_{pp'}[-D_{pp}\nabla^2 \\ &\times (\delta Q^{(0)} - \delta Q^{(2)})^{a'a}_{p'p} + (\delta Q^{(1)} + i\delta Q^{(3)})^{aa'}_{pp'} \\ &\times [-D_{pp'}(\nabla + i\mathbf{a})^2 + C_{pp'}](\delta Q^{(1)} - i\delta Q^{(3)})^{a'a}_{p'p}\} - \bar{\omega}\,\mathrm{tr}[SQ(\mathbf{x})]\Bigg\} \end{aligned} \tag{3.13}$$

where $\mathbf{a} \equiv (2e/\hbar c)\,\mathbf{A}(\mathbf{x})$. Here $\delta Q^{(i)}$, $i = 0, 1, 2, 3$, is the ith component of δQ, i.e., $\delta Q = \sum_{i=0}^{3} \delta Q^{(i)} \otimes \sigma_i$ and to leading order in the disorder γ,

$$\begin{aligned} C_{pp'} &\approx \delta_{pp'} \\ D_{pp'} &\approx (1-\delta_{pp'})\frac{E\hbar^6}{m^3\gamma^2}\frac{2\Gamma^2(d/2)}{d}\left(\frac{2\pi\hbar^3}{mE}\right)^{d-2} \end{aligned} \tag{3.14}$$

From Eqs. (3.13) and (3.14), it follows that the fluctuations for $p = p'$ will be suppressed by the nonvanishing value of $C_{pp'}$. The fluctuations of

$p \neq p'$ are critical. In other words, the fluctuations about the mean field theory are longitudinally massive and transversely massless. We assume that the massive fluctuations can be ignored. We also require that the original symmetry be preserved. Hence, we replace the longitudinal fluctuations by using the condition on the Q field, Eq. (3.5). Using that Q_s is uniform in space, we finally obtain the nonlinear sigma model in the presence of a magnetic field,[13]

$$A[Q] = \frac{1}{t} \int d^d x \, \mathrm{tr}\{(Q^{(0)} + Q^{(2)})(-\nabla^2)(Q^{(0)} - Q^{(2)}) + (Q^{(1)} + iQ^{(3)})[-(\nabla + i\mathbf{a})^2](Q^{(1)} - iQ^{(3)}) - hSQ\} \quad (3.15)$$

with

$$Q^2 = -1, \quad Q^{\mathrm{T}} = Q \quad (3.16)$$

where

$$\frac{h}{t} = \frac{m\omega}{2\hbar^2 \Gamma(d/2)} \left(\frac{mE}{2\pi\hbar^2}\right)^{(d-2)/2}$$

the "temperature" or the disorder coupling constant is $t = 2dm\gamma/E\hbar^d$, and we have scaled the matrix fields by a factor of

$$\frac{m\gamma^{1/2}}{2\hbar^2 \Gamma(d/2)} \left(\frac{mE}{2\pi\hbar^2}\right)^{(d-2)/2}$$

so that the fields are dimensionless. It is easy to show that t is inversely proportional to the Boltzmann diffusivity and that t is dimensionless at $d = 2$.

4. THE FREE ENERGY OR GENERATING FUNCTIONAL

The presence of a magnetic field complicates the renormalization group solution of the nonlinear σ-model. The usual momentum-shell or field-theoretic schemes do not work directly because the propagator involving the magnetic field is not diagonal in momentum space.[15] We note, however, that the theory does not prefer any direction. This implies that the generating functional defined in Eq. (2.1) depends only on the strength of the magnetic field. This fact suggests that one should calculate and renormalize the generating functional.

Our aim is to study the crossover and in particular to study the stability of the pseudo-orthogonal fixed point subject to a small magnetic field. In this section we use perturbative methods to derive a generating functional density containing information on both symmetries. We introduce a renormalization procedure whereby it is possible to obtain both pseudo-orthogonal and pseudo-unitary fixed points. The crossover behavior is then simply obtained. We expand the effective action functional, Eq. (3.15), in terms of the massless transverse components of the fluctuation fields Q to establish a loop expansion for the generating functional density. We will also generalize the dimensional regularization to the case where a magnetic field is applied. As usual, the constraints given by Eq. (3.16) are satisfied by parametizing the matrix Q by

$$Q=\begin{pmatrix} -i(1+qq^{\mathrm{T}})^{1/2} & q^{\mathrm{T}} \\ q & i(1+q^{\mathrm{T}}q)^{1/2} \end{pmatrix} \tag{4.1}$$

q in a quaternion representation is

$$q=\sum_{i=0}^{3} q^{(i)}\otimes\sigma_i \tag{4.2}$$

where $q^{(i)}$, $i=0, 1, 2, 3$, is an $n\times n$ real matrix. Further, we define

$$\theta\equiv q^{(0)}+iq^{(2)}, \qquad \psi\equiv q^{(1)}+iq^{(3)} \tag{4.3}$$

For the one-loop calculation, we only need to expand Eq. (3.15) up to $O(q^4)$. Using Eq. (4.3), it is straightforward to obtain,

$$A=C+A_0+A_I \tag{4.4}$$

where

$$C\equiv\frac{4nhV}{t} \tag{4.5}$$

is a constant, with V the volume of the system. In Eq. (4.4),

$$A_0\equiv\frac{2}{t}\int d^dx\,\mathrm{tr}[\theta(-\nabla^2+h)\,\theta^\dagger+\psi(-\nabla^2-2i\mathbf{a}\cdot\nabla+\mathbf{a}^2+h)\,\psi^\dagger] \tag{4.6}$$

is the "free part" of the action functional and A_I is the interaction part.

To implement the renormalization procedure, we calculate the generating functional density per unit volume per replicated system defined as [cf. Eq. (2.1)]

$$f\equiv\frac{F_n}{4nV}=-\frac{1}{4nV}\ln\left(\int D\theta\,D\theta^\dagger\,D\psi\,D\psi^\dagger\,e^{-A_0-A_I-C}\right) \tag{4.7}$$

It is not difficult to show that

$$-\ln\left(\int D\theta\, D\theta^\dagger\, D\psi\, D\psi^\dagger\, e^{-A_0}\right)=n^2[g(h,\alpha)+g(h,0)] \tag{4.8}$$

where $\alpha\equiv eB/\hbar c$ and

$$g(h,\alpha)\equiv\int d^d\mathbf{x}\,\langle\mathbf{x}|\ln(-\nabla^2-2i\mathbf{a}\cdot\nabla+\mathbf{a}^2+h)\,|\mathbf{x}\rangle \tag{4.9}$$

Thus, we have the following loop expansion for the generating functional density:

$$f=\frac{h}{t}+\frac{n}{4V}\,[g(h,\alpha)+g(h,0)]+\frac{1}{4nV}\langle A_I\rangle+\cdots \tag{4.10}$$

where $\langle\cdot\rangle$ denotes an average with weight $\exp(-A_0)$.

For the calculation of $\langle A_I\rangle$ we need correlation functions of the fields. Using that A [cf. Eq. (4.6)] is a Gaussian statistical weight, we have

$$\langle\theta^*_{ij}(\mathbf{x})\,\theta_{lm}(\mathbf{y})\rangle=\frac{t}{2}\,\delta_{il}\delta_{jm}f_0(\mathbf{x},\mathbf{y}) \tag{4.11a}$$

$$\langle\psi^*_{ij}(\mathbf{x})\,\psi_{lm}(\mathbf{y})\rangle=\frac{t}{2}\,\delta_{il}\delta_{jm}f_\alpha(\mathbf{x},\mathbf{y}) \tag{4.11b}$$

$$\begin{aligned}\langle\theta_{ij}(\mathbf{x})\,\theta_{lm}(\mathbf{y})\rangle&=\langle\theta_{ij}(\mathbf{x})\,\psi^*_{lm}(\mathbf{y})\rangle=\langle\theta_{ij}(\mathbf{x})\,\psi_{lm}(\mathbf{y})\rangle\\&=\langle\psi_{ij}(\mathbf{x})\,\psi_{lm}(\mathbf{y})\rangle=0\end{aligned} \tag{4.11c}$$

where

$$f_\alpha(\mathbf{x},\mathbf{y})=\langle\mathbf{x}|\,(-\nabla^2-2i\,\mathbf{a}\cdot\nabla+\mathbf{a}^2+h)^{-1}\,|\mathbf{y}\rangle \tag{4.12}$$

Further, because the statistical weight is Gaussian, we need only these two-point correlation functions. Therefore, the averaging process becomes summation over all possible products of pair "contractions," each of which is replaced by functions f_α, f_0 and their derivatives. As for cases where partial derivatives appear, it is obvious that we should distribute each derivative in all possible legitimate ways. In particular, at the one-loop order we have at most two positions for each derivative. In this way, and by summing over all indices, we obtain

$$\langle A_I\rangle=\frac{n^2Vhtf_0f_\alpha}{4} \tag{4.13}$$

Here, $f_\alpha\equiv f_\alpha(x,x)$, $f_0\equiv f_0(x,x)$.

To calculate f_α in d dimension, we generalize dimensional regularization to the case where the magnetic field is applied. For $d \geqslant 2$ we use

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_d} \right) \tag{4.14}$$

$$\alpha = \alpha(-x_2, x_1, 0, ..., 0) \tag{4.15}$$

Applying the path integral technique,[14] it is easily found that

$$f_\alpha = f_0 + \frac{1}{(4\pi)^{d/2}} \frac{\partial J(h, \alpha)}{\partial h} \tag{4.16a}$$

$$f_0 = \frac{h^{(d-2)/2}}{(4\pi)^{d/2}} \Gamma\left(\frac{2-d}{2} \right) \tag{4.16b}$$

where

$$J(h, \alpha) \equiv \int_0^\infty dx \left(\frac{1}{x} - \frac{2\alpha}{\sinh 2\alpha x} \right) e^{-hx - (d/2) \ln x} \tag{4.17}$$

Note that

$$g(h, \alpha) - g(h, 0) = -V \int_h^\infty (f_\alpha - f_0)\, dh \tag{4.18a}$$

Using that J vanishes as h tends to infinity, we find

$$g(h, \alpha) = g(h, 0) + \frac{VJ(h, \alpha)}{(4\pi)^{d/2}} \tag{4.18b}$$

By direct calculation, we obtain

$$g(h, 0) = \frac{2Vh^{d/2}}{d(4\pi)^{d/2}} \Gamma\left(\frac{2-d}{2} \right) \tag{4.18c}$$

Here we have dropped a term that vanishes in dimensional regularization.

Therefore, up to the one-loop order, we obtain the generating functional density which contains both pseudo-orthogonal and pseudo-unitary symmetries, and from which we should be able to obtain the fixed points for both symmetries as well as crossover between them. Scaling the generating functional density by a factor $(4\pi)^{-d/2}$, we have

$$f = \frac{h}{T} + \frac{nh^{d/2}}{d} \Gamma\left(\frac{2-d}{2} \right) + \frac{n}{4} J(h, \alpha) + \frac{nh^{d-1}T}{16} \left[\Gamma\left(\frac{2-d}{2} \right) \right]^2 + \frac{nh^{d/2}T}{16} \Gamma\left(\frac{2-d}{2} \right) \frac{\partial J(h, \alpha)}{\partial h} + O(T^2) \tag{4.19}$$

where the disorder is scaled by $(4\pi)^{d/2}$, i.e., $T = t(4\pi)^{-d/2}$. In terms of the generalized ζ function,

$$J(h, \alpha) = -\frac{nh^{d/2}}{d} \Gamma\left(\frac{2-d}{d}\right) C\left(\frac{h}{\alpha}, d\right) \tag{4.20a}$$

$$\frac{\partial J(h, \alpha)}{\partial h} = -h^{(d-2)/2} \Gamma\left(\frac{2-d}{2}\right) C\left(\frac{h}{\alpha}, d-2\right) \tag{4.20b}$$

with

$$C\left(\frac{h}{\alpha}, d\right) = \begin{cases} 1 + \frac{d}{2}\left(\frac{4\alpha}{h}\right)^{d/2} \zeta\left(\frac{2-d}{2}, \frac{h}{4\alpha} + \frac{1}{2}\right) & (\alpha \text{ finite}) \\ 1 \quad \text{for} \quad \frac{1}{d-2} \ll \alpha \to \infty \end{cases} \tag{4.20c}$$

$\alpha = 0$ is the usual pseudo-orthogonal case. For $\alpha \to \infty$ the magnetic field dominates, in which case the field $Q^{(1)} + iQ^{(3)}$ and its Hermitian conjugation in Eq. (3.15) are suppressed. However, $Q^{(0)} + Q^{(2)}$ is a complex field. Hence, this is the case of pseudo-unitary symmetry. Therefore, we obtain the generating functional densities for both the orthogonal and the unitary cases directly from Eq. (4.19). The two limiting cases are results obtained by, for example, ref. 17, up to the one-loop order,

$$f_{(\mathrm{O})} = \frac{h}{T} + \frac{nh^{d/2}}{d} \Gamma\left(-\frac{\varepsilon}{2}\right) + \frac{nh^{d-1}T}{16} + \left[\Gamma\left(-\frac{\varepsilon}{2}\right)\right]^2 + O(T^2) \tag{4.21a}$$

$$f_{(\mathrm{U})} = \frac{h}{T} + \frac{nh^{d/2}}{2d} \Gamma\left(-\frac{\varepsilon}{2}\right) + O(T^2) \tag{4.21b}$$

The factor of two difference in the second terms of the two cases is due to the fact that in the unitary case the number of replicated fields is half of that in the orthogonal case.

5. RENORMALIZATION AND THE CROSSOVER BEHAVIOR

The renormalizability of the nonlinear σ-model in the presence of position-independent perturbing terms or fields has been proven by Brézin *et al.*[18] The proof does not directly apply to the case where the field depends on position. We assume here that because the magnetic field is a physical field, it introduces at most one more renormalization constant into the theory. To one-loop order, we show that the magnetic field actually introduces no additional renormalization constants into the theory and that the magnetic field has its naive scaling dimension.

We perform the renormalization of the generating functional using a generalized minimal subtraction (GMS) scheme.[8] The generating functional (4.19) is in general divergent (as $\varepsilon \equiv d-2 \to 0$) at every order in the disorder except for the first term. We denote the unrenormalized or bare quantities by $(f_B, T_B, h_B, \alpha_B)$. The renormalized generating functional and other parameters are related to the unrenormalized ones by the following relations:

$$f(\mu, T, h, \alpha) = f_B(T_B, h_B, \alpha_B) \tag{5.1}$$

$$T = Z_T^{-1} \mu^{\varepsilon} T_B \tag{5.1a}$$

$$h = Z_h^{-1} \mu h_B \tag{5.1b}$$

$$\alpha = Z_\alpha^{-1} \mu \alpha_B \tag{5.1c}$$

and the powers of μ in Eqs. (5.1) are chosen such that T, h, and α are dimensionless. Z_T, Z_h, are Z_α are renormalization constants.

After the renormalization procedure, all divergences (as $\varepsilon \to 0$) are absorbed into the renormalization constants. Expanding the renormalization constants in powers of T, we have

$$\frac{Z_h}{Z_T} = 1 + e_1 T + e_2 T^2 + O(T^3) \tag{5.2}$$

$$Z_h = 1 + bT + O(T^2) \tag{5.3}$$

$$Z_\alpha = 1 + cT + O(T^2) \tag{5.4}$$

where the expansion coefficients are in general functions of h and α, and to be determined by the MS scheme. The generating functional density f is not expected to be divergent as $\varepsilon \to 0$. This immediately yields, for small h,

$$c = 0 \tag{5.5}$$

$$e_1 = \frac{n}{\varepsilon}\left[1 - \frac{C(h/\alpha, \varepsilon + 2)}{2}\right] \tag{5.6}$$

$$e_2 = \frac{nb}{\varepsilon}\left[1 - \frac{C(h/\alpha, \varepsilon)}{2}\right]\left[1 + \frac{\varepsilon}{2}(\gamma + \ln h)\right] - \frac{n}{4\varepsilon^2}\left[1 - C\left(\frac{h}{\varepsilon}, \varepsilon\right)\right](1 + \varepsilon\gamma) \tag{5.7}$$

Using that the generating functional is invariant under the renormalization transformation, we can formally derive the renormalization group flow equations. With e^l the RG length scale factor, the flow equations in the parameter space spanned by T, $h/T \equiv \Omega$, and α are

$$\frac{dT}{dl} = -\varepsilon T[1-(b-e_1)T] + O(T^3) \tag{5.8}$$

$$\frac{d\alpha}{dl} = 2\alpha + O(T^2) \tag{5.9}$$

$$\frac{d\Omega}{dl} = \Omega(d+\gamma_\Omega) \tag{5.10}$$

where the anomalous dimension of Ω

$$\begin{aligned}\gamma_\Omega = T&\left[\varepsilon\left(e_1 - \Omega\frac{\partial e_1}{\partial\Omega}\right) - 2\left(\alpha\frac{\partial e_1}{\partial\alpha} + \Omega\frac{\partial e_1}{\partial\Omega}\right)\right]\\ &+ T^2\left\{\varepsilon\left(2e_2 - be_1 - \Omega\frac{\partial e_2}{\partial\Omega}\right) - 2\left(\alpha\frac{\partial e_2}{\partial\alpha} + \Omega\frac{\partial e_2}{\partial\Omega}\right)\right.\\ &\left.+ \alpha\frac{\partial e_1^2}{\partial\alpha} + \Omega\frac{\partial e_1}{\partial\Omega}\left[2\left(e_1 + \Omega\frac{\partial e_1}{\partial\Omega} + \alpha\frac{\partial e_1}{\partial\alpha}\right) + \varepsilon\Omega\frac{\partial e_1}{\partial\Omega}\right]\right\} + O(T^3)\end{aligned} \tag{5.11}$$

has to be finite as well in the limit $\varepsilon \to 0$. Thus we find

$$b = \frac{1}{2\varepsilon}\frac{1 - C(\Omega T/\alpha, \varepsilon)}{1 - C(\Omega T/\alpha, \varepsilon) + C(\Omega T\alpha, \varepsilon+2)/2} + O(1) \tag{5.12}$$

Therefore, by taking the limit $n \to 0$, the RG flow equations are obtained:

$$\frac{dT}{dl} = -\varepsilon T\left[1 - \frac{T}{2\varepsilon}\frac{1 - C(\Omega T/\alpha, \varepsilon)}{1 - C(\Omega T/\alpha, \varepsilon) + C(\Omega T/\alpha, \varepsilon+2)/2}\right] + O(T^3) \tag{5.13}$$

$$\frac{d\alpha}{dl} = 2\alpha + O(T^2) \tag{5.14}$$

$$\frac{d\Omega}{dl} = (2+\varepsilon)\Omega \tag{5.15}$$

Equation (5.15) is exact in the $n \to 0$ limit.

Because the theory is symmetric in α, i.e., in the sign of the magnetic field strength, we assume that α is positive. Defining

$$\chi \equiv \frac{1}{1 + \Omega T/\alpha} \tag{5.16}$$

we have

$$\frac{dT}{dl} = -\varepsilon T \left\{ 1 - \frac{T}{2\varepsilon} \frac{1 - C[(1-\chi)/\chi, \varepsilon]}{1 - C[(1-\chi)/\chi, \varepsilon] + C[(1-\chi)/\chi, \varepsilon + 2]/2} \right\} \tag{5.17}$$

$$\frac{d\chi}{dl} = -\chi(1-\chi) \frac{T}{2} \frac{1 - C[(1-\chi)/\chi, \varepsilon]}{1 - C[(1-\chi)/\chi, \varepsilon] + C[(1-\chi)/\chi, \varepsilon + 2]/2} \tag{5.18}$$

$$\frac{d\Omega}{dl} = (2+\varepsilon)\Omega \tag{5.19}$$

We see that Eq. (5.19) is decoupled from Eqs. (5.17) and (5.18). This allows us to project the flow onto a plane with a given value of Ω. In the parameter space $P = (T, \chi)$ two fixed points are found:

$$P_{(\mathrm{O})} = (2\varepsilon, 0) \tag{5.20}$$

$$P_{(\mathrm{U})} = (m\varepsilon^{1/2}, 1) \tag{5.21}$$

Here, the coefficient m can be determined by a higher-order (in disorder) calculation. $T = 0$ is a fixed line which is not interesting in the present analysis.

The nontrivial fixed poins $P_{(\mathrm{O})}$ and $P_{(\mathrm{U})}$ are the known results for the pseudo-orthogonal and the pseudo-unitary symmetries. For the pseudo-orthogonal case, $\chi = 0$ or $\alpha = 0$. For the pseudo-unitary case, $\chi = 1$ or $\alpha = \infty$. The advantage of our formalism is that the crossover behavior becomes obvious.

Near the orthogonal fixed point $P_{(\mathrm{O})}$ we find that to the lowest order in loop expansion,

$$|t(b)| = \mathrm{b}^{\varepsilon} \, |t| \tag{5.22}$$

$$\alpha(b) = b^2 \alpha \tag{5.23}$$

where b is a length scale and t is the deviation of T from its pseudo-orthogonal fixed point value 2ε. In general, the ε in Eq. (5.22) is $1/\nu$, where ν is the localization length exponent. The generating functional density, correlation (or localization) length, and the conductivity have the following scaling forms around the orthogonal fixed point $P_{(\mathrm{O})}$ (at zero Ω):

$$f(|t|, \alpha) = |t|^{\nu(2+\varepsilon)} \, Y_1\left(\frac{\alpha}{|t|^{2/\nu}}\right) \tag{5.24a}$$

$$\xi(|t|, \alpha) = |t|^{-\nu} \, Y_2\left(\frac{\alpha}{|t|^{2/\nu}}\right) \tag{5.24b}$$

$$\sigma(|t|, \alpha) = |t|^{\nu\varepsilon} \, Y_3\left(\frac{\alpha}{|t|^{2/\nu}}\right) \tag{5.24c}$$

which implies that the crossover exponent is

$$\phi = \frac{2}{\nu}[1 + O(\varepsilon^2)] \tag{5.25}$$

This is in agrreement with other work.[5-7,9] Note that the 2 in Eqs. (5.23) and (5.25) is trivial and follows from dimensional analysis. Our nontrivial result is that the order-ε correction to the 2 vanishes.[3] In their phenomenological analysis, Khmel'nitskii and Larkin[9] conclude that 2 is the exact result, which is also suggested by Biafore *et al.*,[7] using a gauge invariance argument. In any case, the magnetic field is a relevant perturbation near the orthogonal fixed point. The magnetic field causes the crossover by driving the system away from the orthogonal fixed point to the unitary fixed point $P_{(\mathrm{U})}$.

Note that $|t|$ is the difference between the inverse of the bare conductivity and its critical value and consequently it is the dimensionless distance from the mobility edge. For $\alpha = 0$, we find

$$f \sim |E - E_c|^{\nu(\varepsilon+2)} \tag{5.26a}$$

$$\xi \sim |E - E_c|^{-\nu} \tag{5.26b}$$

$$\sigma \sim |E - E_c|^s, \qquad s = \nu\varepsilon \tag{5.26c}$$

To lowest order in ε, $\nu = \varepsilon^{-1}$.

Alternatively, we also have

$$f \sim \alpha^{(2+\varepsilon)2} Z_1(|t|\, \alpha^{-1/2\nu}) \tag{5.27a}$$

$$\xi \sim \alpha^{-1/2} Z_2(|t|\, \alpha^{-1/2\nu}) \tag{5.27b}$$

$$\sigma \sim \alpha^{\varepsilon/2} Z_3(|t|\, \alpha^{-1/2\nu}) \tag{5.27c}$$

Using that $|t| = 0$ corresponds to the mobility edge for the system without the magnetic field, and that for $\alpha \neq 0$, this does not represent the mobility edge any more, we then expect $Z_3(0)$ to be finite. Therefore, the mobility edge must be shifted. The new mobility edge is determined by a finite value of the argument of Z, i.e., at

$$|t|\, \alpha^{-1/2\nu} = \Lambda \tag{5.28}$$

where Λ is an undetermined number. The critical disorder is shifted by

$$T_c(\alpha) = T_c(1 + \Lambda\alpha^{1/2\nu}) \tag{5.29}$$

[3] For the case where there is a crossover due to a position-independent external field, a similar result is found.[3,4]

This is also in agreement with the phenomenological analysis of Khmel'nitskii and Larkin.[9]

The exact position of $P_{(U)}$ can be determined by doing a two-loop calculation. Previous work finds that the coefficient in Eq. (5.21) is $m = 2^{-1/2}$. With $\chi = 1$, we linearize the flow equation for temperature near $P_{(U)}$ and find

$$\frac{dt}{dl} = 2\varepsilon t + O(t^2) \tag{5.30}$$

Equation (5.30) implies the scaling behavior given by Eq. (5.26c) with $v = 1/2\varepsilon$.

It is easy to verify that $d\chi/dT \to +\infty$ as $T = 2\varepsilon$, and $\chi \to 0^+$. Therefore, the RG flow runs away from the fixed point $P_{(O)}$ vertically. The details of the flow beneath the fixed point $P_{(U)}$ rely on the higher-order calculation.

6. DISCUSSION

In summary, we have set up a field-theoretic formalism and derived a generalized nonlinear sigma model which contains both pseudo-orthogonal and pseudo-unitary symmetries to study the critical and crossover behavior in the vicinity of the mobility edge for systems when a magnetic field is applied. By applying a dimensional regularization and minimal subtraction scheme, we renormalized the generating functional. We then obtained the renormalization-group flow equations. These flow equations unify the two symmetries, i.e., they contain fixed points with pseudo-unitary and pseudo-orthogonal symmetries, so that we recovered the critical behavior for these two symmetries which have been studied separately previously. The cross-over exponent near the orthogonal field is relevant and it drives the system away from the orthogonal fixed point to the unitary fixed point. The RG flow diagram has been obtained.

In real electronic systems the effects of electron–electron interaction are unavoidable. Recent work strongly suggests that it is impossible to properly describe the metal–insulator transition without including these interaction effects (see, e.g., ref. 19). However, it is probable that the pure localization phase transition can be observed in systems where light or acoustic waves are localized. In two dimensions the crossover phenomena discussed in this paper are analogous to the localization of third-sound waves on a disordered substrate in the presence of a uniform superflow.[4]

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REFERENCES

1. P. Lee and T. V. Ramakrishnan, *Rev. Mod. Phys.* **57**:287 (1985).
2. F. Wegner, *Z. Phys. B* **35**:307 (1979).
3. F. Wegner, *Nucl. Phys. B* **270[FS16]**:1 (1986).
4. S. M. Cohen, J. Machta, T. R. Kirkpatrick, and C. A. Condat, *Phys. Rev. Lett.* **58**:795 (1987).
5. A. Houghton, A. J. McKane, and H. A. Cerdeira, *J. Stat. Phys.* **48** (1987).
6. R. Oppermann, *J. Phys.* (Paris) **45**:L1161 (1984); D. Belitz, *Solid State Comm.* **52**:989 (1984).
7. M. Biafore, C. Castellani, and G. Kotliar, *Nucl. Phys. B* **340**:617 (1990).
8. D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984); see also D. J. Amit, Y. Y. Goldschmidt, and L. Peliti, *Ann. Phys.* **116**:1 (1978).
9. D. E. Khmel'nitskii and A. I. Larkin, *Solid State Commun.* **39**:1069 (1981).
10. F. Wegner, *Nucl. Phys. B* **316**:1 (1989).
10. A. M. M. Pruisken and L. Schafer, *Nucl. Phys. B* **200[FS4]**:20 (1982).
12. K. B. Efetov, A. I. Larkin, and D. E. Kheml'nitskii, *Sov. Phys. JETP* **52**(3):568 (1980).
13. A. M. M. Pruiskin, *Nucl. Phys. B* **235[FS11]**:277 (1984).
14. A. J. McKane and M. Stone, *Ann. Phys.* (N.Y.) **131**:36 (1981).
15. D. Schmeltzer, *Phys. Rev. B* **36**:5548 (1987).
16. R. P. Feynman and A. R. Hibbs, *Quantum Mechanis and Path Integrals* (McGraw-Hill, New York, 1965).
17. D. Höf and F. Wegner, *Nucl. Phys. B* **275**:561 (1986).
18. E. Brézin, J. Zinn-Justin, and J. C. Le Guillou, *Phys. Rev. D* **14**:2615 (1980).
19. C. Di Castro, in *Anderson Localization*, M. T. Ando and H. Fukuyama, eds. (Springer, New York, 1989).